

Implementation of multipartite unitary operations with limited resources

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A general method for implementing weakly entangling multipartite unitary operations using a small amount of entanglement and classical communication is presented. For the simple Hamiltonian $\sigma_z \otimes \sigma_z$ this method requires less entanglement than previously known methods. In addition, compression of multiple operations is applied to reduce the average communication required.

I. INTRODUCTION

Much of quantum information processing relies on the ability to perform operations between subsystems. In the case where these subsystems do not directly interact, it is necessary to achieve the operation by indirect means. For example, one could use a third subsystem which interacts with both, or use entanglement. In the case where the operation is only weakly entangling, it would be useful to be able to implement it using only a small amount of entanglement.

A SWAP operation may be applied using two teleportations [1], requiring two ebits of entanglement and two bits of communication in each direction. Similarly, a CNOT may be applied using one ebit and one bit of communication in each direction [2]. For more general two-qubit unitaries, weakly entangling unitaries may be implemented using a small amount of entanglement [3]. Ref. [3] gives a scheme to implement operations of the form $e^{-i\alpha\sigma_x\otimes\sigma_x}$ using average entanglement of 5.9793α . This may also be applied to more general two-qubit unitaries. Alternative schemes for operations of this form, but allowing a significant probability of failure are presented in Refs. [4, 5]. A scheme for Hamiltonians which are a tensor product of self-inverse operators was given in Ref. [6].

Here we present a general method of implementing weakly entangling multipartite unitary operations using a small amount of entanglement or communication. The situation considered is that there are N parties, each with a number of subsystems, and operations performed by the parties are assumed to have negligible cost. The unitary we wish to implement acts upon one subsystem in possession of each of the parties, and we quantify the entanglement and classical communication between the parties.

This paper proceeds as follows. The general problem of implementing operations using entanglement and communication is presented in Sec. II. Then an efficient method for implementing evolution under multipartite Hamiltonians of the form $\sigma_z^{\otimes N}$ is presented in Sec. III. A method of using compression over multiple operations to reduce the communication required is given in Sec. IV. A method of generalising these results to tensor product Hamiltonians is given in Sec. V, and further generalisations are given in Sec. VI. Conclusions are given in Sec. VII.

II. GENERAL METHODS

The most general method of implementing an operation using entanglement and classical communication may be described as follows. The N parties share an entangled resource state, and perform many rounds of the following process. Each party performs a local unitary followed by a partial measurement. The parties then communicate classical information about the measurement results, and repeat the process with local unitaries and measurements possibly depending on the transmitted information.

In general we consider many implementations of U , and regard the scheme as successful if the minimum fidelity of the output state approaches one as the number of copies of U approaches infinity. We may define a sufficient rate vector \vec{R} as a vector of measures of multipartite entanglement and classical communication such that, for all $\epsilon > 0$ there exists an n such that n implementations of U may be performed using average entanglement and classical communication no larger than the components of \vec{R} , and minimum fidelity at least $1 - \epsilon$.

The scheme of Cirac, Dür, Kraus and Lewenstein (CDKL) [3] is of this form. Joint measurements between an entangled resource state and the target system are made. The correct measurement result means that the operation is correctly performed. If failure occurs, another resource state is used, and measurement success corrects for the previous failure and correctly implements the unitary. For repeated failure this process is repeated until the operation which is required may be performed locally.

In the following we improve on the CDKL scheme in three ways:

1. We reduce the entanglement consumption.
2. We reduce the communication in most directions to be the same as the entanglement consumption.
3. We apply the scheme to more general Hamiltonians.

III. ENTANGLEMENT CONSUMPTION

First we give an improved scheme which reduces the entanglement consumed. We consider the Hamiltonian $\sigma_z^{\otimes N}$ for N parties, so the unitary we wish to apply is $U(\alpha) = \exp(i\alpha\sigma_z^{\otimes N})$. This is the multipartite generalisation of the Hamiltonian considered by CDKL. (We use

σ_z here rather than σ_x , but these are equivalent under local unitaries.)

The N parties use the resource state

$$|\psi(\beta)\rangle = \cos(\beta)|0\rangle^{\otimes N} + i \sin(\beta)|1\rangle^{\otimes N}, \quad (1)$$

then apply a four step process. In the following we use the “stator” formalism, introduced in Ref. [7]. A stator is a hybrid state-operator object, and acts upon a state by applying its operator component to the state, and appending its state component. That is, the stator $|\phi\rangle \otimes U$ acts on $|\psi\rangle$ as

$$(|\phi\rangle \otimes U)|\psi\rangle = |\phi\rangle \otimes (U|\psi\rangle). \quad (2)$$

Linearity is used to obtain the action of staters which are sums of terms of the form $|\phi\rangle \otimes U$. The four steps are as follows:

Step 1. Each party applies a controlled- Z operation on the target system. This yields the stator

$$\cos(\beta)|0\rangle^{\otimes N} \otimes \mathbb{1}^{\otimes N} + i \sin(\beta)|1\rangle^{\otimes N} \otimes \sigma_z^{\otimes N} \quad (3)$$

Step 2. Parties 1 to $N - 1$ apply Hadamard operations followed by computational basis measurements on their component of the resource state. The resulting stator is

$$\cos(\beta)|0\rangle \otimes \mathbb{1}^{\otimes N} \pm i \sin(\beta)|1\rangle \otimes \sigma_z^{\otimes N}, \quad (4)$$

where the sign is $+$ if the number of measurement results equal to 1 is even, and $-$ otherwise.

Step 3. Party N performs a correction on their component of the resource state based on the measurement results. If the number of measurement results equal to 1 is odd, then they perform a σ_z operation on the remaining portion of the resource state. We then obtain the stator with the plus in Eq. (4).

Step 4. Party N applies a projection measurement on their component of the resource state. Projection onto the state $|\psi\rangle = \cos(\gamma)|0\rangle + \sin(\gamma)|1\rangle$ yields the unitary operation proportional to

$$\cos(\beta)\cos(\gamma) \otimes \mathbb{1}^{\otimes N} + i \sin(\beta)\sin(\gamma) \otimes \sigma_z^{\otimes N}. \quad (5)$$

For failure, projection onto the state $|\psi^\perp\rangle = \sin(\gamma)|0\rangle - \cos(\gamma)|1\rangle$ is obtained, giving an operation proportional to

$$\cos(\beta)\sin(\gamma) \otimes \mathbb{1}^{\otimes N} - i \sin(\beta)\cos(\gamma) \otimes \sigma_z^{\otimes N}. \quad (6)$$

The measurement in Step 4 is chosen such that, for the correct measurement result, the unitary $U(\alpha)$ is applied. This will be obtained provided

$$\tan(\beta)\tan(\gamma) = \tan(\alpha). \quad (7)$$

For the incorrect measurement result, the unitary is of the form $U(\alpha')$, but with $\alpha' \neq \alpha$. To obtain the correct result, we now wish to implement the unitary $U(\alpha - \alpha')$. Combined with the previous incorrect unitary $U(\alpha')$, this

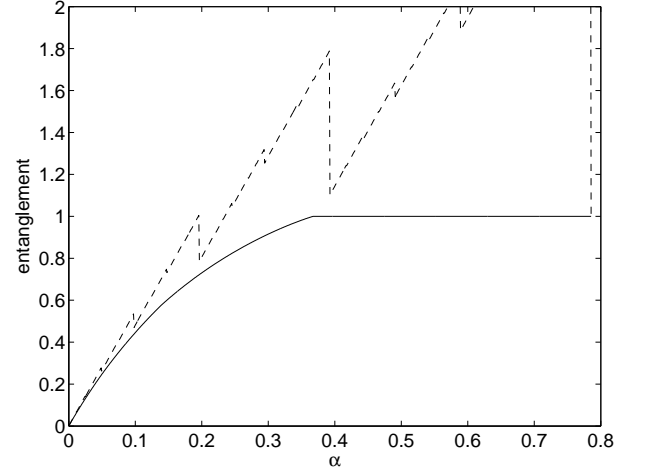


FIG. 1: The average entanglement consumed to implement the operation $U(\alpha)$ as a function of α . The case where the resource states have been numerically optimised is shown as the solid line, and the case for the scheme of Ref. [3] is shown as the dashed line.

will result in the desired $U(\alpha)$. To achieve this, one requires a new resource state $|\psi(\beta')\rangle$ (where β' is not in general equal to β) and repeats the process.

If the incorrect measurement result is obtained repeatedly, one can obtain the correct unitary by using the resource state $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$. One can deterministically perform any unitary of the form $U(\alpha)$ with this resource state. To see this, consider steps 1 to 4 with $\beta = \pi/2$ and $\gamma = \alpha$. Then, in the case of failure one simply applies the local unitary $U(\pi/2)$.

In the following the terminology “stage” is used to mean steps 1 to 4. The notation β_l is used for the parameter for the entangled resource state used in stage l (i.e. after $l - 1$ failures). In addition, α_l is used for the parameter for the unitary we wish to implement in stage l . That is, $\alpha_1 = \alpha$, $U(\alpha_2)$ is the unitary we require for the correction after one failure, and so forth. We also use L for the maximum total number of stages.

If $N = 2$, $\alpha = \pi/2n$ and $\beta_l = \alpha_l$, this method is similar to that of CDKL. At each stage the probability of success is equal to $1/2$, and for repeated failure we need to implement $U(\pi/2)$, which may be performed locally. The resource state is different from that used by CDKL, but the average entanglement consumed is identical.

To improve upon the entanglement consumption we adjust the entanglement of the resource states. By using resource states with larger entanglement, the probability for success is increased, and the average entanglement consumed is reduced. It is straightforward to numerically optimise for the resource states that minimise the average entanglement consumed for given α . The resulting average entanglement is plotted in Fig. 1 as a function of α .

The entanglement consumed for the CDKL scheme is also shown in Fig. 1. The CDKL scheme is explicitly

given for values of α of the form $\alpha = \pi/2^n$. However, it is trivial to apply it to other values of α by making an expansion $\alpha = \sum_n a_n \pi/2^n$, with $a_n \in \{0, 1\}$. The entanglement consumed may then be determined by summing it over that for the individual terms. This is how the values shown in Fig. 1 were determined.

For all values of α (except the trivial $\alpha = 0$) the entanglement consumed for the numerically optimised scheme is less than that for the CDKL scheme. For small α the entanglement consumed is approximately 5.6418α , as compared to 5.9793α for the CDKL scheme.

To show that the entanglement consumed is approximately 5.6418α in the limit of small α , we may use the following approach. First we determine the entanglement required to implement $U(A)$ for values of A over a range of a factor of two. Here we take $A \in [\pi/2^{20}, \pi/2^{19}]$. Now for values of α below $\pi/2^{20}$, we select an A in the range $[\pi/2^{20}, \pi/2^{19}]$, such that $\alpha = A/2^n$. Then, rather than numerically optimising for the best intermediate entangled states, we select the intermediate entangled states with $\beta_l = \alpha 2^{l-1}$ for $1 \leq l \leq n$, so $\beta_l = \alpha_l$. If a successful measurement has not been obtained before stage $n+1$, then the correction required is $U(A)$; this may be achieved using the numerically optimised scheme.

The average entanglement consumed is

$$\mathcal{E}(\alpha) = \mathcal{E}(A)2^{-n} + \sum_{l=1}^n 2^{1-l} E(\alpha_l), \quad (8)$$

where we use $\mathcal{E}(\alpha)$ for the entanglement consumed to implement $U(\alpha)$, and $E(\alpha)$ for the entanglement of the resource state $|\psi(\alpha)\rangle$. We have $E(\alpha) = h[\sin^2(\alpha)]$ where h is the binary entropy function $h(p) = -p \log_2(p) - (1-p) \log_2(1-p)$.

This entanglement measure is the entropy of the reduced density operator for one subsystem. This is a consistent entanglement measure for Schmidt decomposable multipartite pure states. It is possible to apply similar methods to the bipartite case to perform entanglement concentration and dilution between these states and the standard states $(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})/\sqrt{2}$ [8].

The first term in Eq. (8) is the probability for n failures (2^{-n}) multiplied by the entanglement required to implement $U(A)$. The sum is the entanglement of the state with $\beta = \alpha_j$ for steps 1 to n multiplied by the probability. Eq. (8) gives the upper bound

$$\frac{\mathcal{E}(\alpha)}{\alpha} \leq \frac{\mathcal{E}(A) + \sum_{k=1}^{\infty} 2^k E(A 2^{-k})}{A}. \quad (9)$$

The expression on the right-hand side (RHS) of Eq. (9) is independent of n , and only depends on A . This expression is plotted as a function of A in Fig. 2 (the irregularity appears to be due to finite precision). It can be seen that this expression does not exceed 5.6418 for this range of A . Thus we find that, in the limit of small α , the entanglement consumed need not exceed 5.6418α .

This proof is not fully rigorous because it is not possible to determine the RHS of Eq. (9) for all values of A

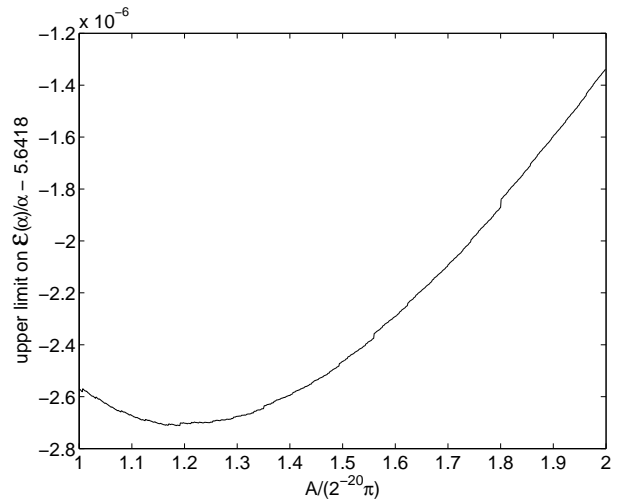


FIG. 2: Upper limit on $\mathcal{E}(\alpha)/\alpha$ [the RHS of Eq. (9)] minus 5.6418 as a function of A .

in this range. For rigour, note that α may be expanded as $\alpha = \sum_n a_n \pi/2^n$, and the simulation of $U(\alpha)$ may be achieved by simulation of the appropriate $U(\pi/2^n)$. In practice, this method is more complicated, however.

Another issue is that the above analysis is in terms of the expectation value of the entanglement consumed, given that one is provided with entangled states on demand. Alternatively we may consider many implementations of the unitary with a fixed set of entangled resource states. This is consistent with the definition given in Sec. II.

Let $p(l)$ be the probability of requiring the entangled resource state in stage l (i.e. the probability of $l-1$ failures). Now, for all $\epsilon, \delta > 0$, there exists an M such that the probability of the number of entangled resource states required in stage l exceeding $Mp(l)(1+\delta)$ is no more than ϵ/L . Therefore, let us consider M implementations of $U(\alpha)$ with $\lfloor Mp(l)(1+\delta) \rfloor$ copies of resource state $|\psi(\beta_l)\rangle$ at each stage. Then the total entanglement does not exceed $M\mathcal{E}(\alpha)(1+\delta)$.

In the case that the number of resource states exceeds that provided in one of the stages, let the output state be ρ_{fail} . Otherwise the correct output state $|\phi\rangle$ is obtained. After discarding the measurement results, the average output state is of the form

$$\rho = p_{\text{fail}}\rho_{\text{fail}} + (1 - p_{\text{fail}})|\phi\rangle\langle\phi|. \quad (10)$$

Because the total probability of requiring more than the number of resource states provided in stage l is no more than ϵ/L , the total probability of failure does not exceed ϵ . Therefore the fidelity must be at least $1 - \epsilon$. Thus we find that, given sufficiently large M , the average entanglement consumed is arbitrarily close to $\mathcal{E}(\alpha)$ with fidelity arbitrarily close to 1.

IV. CLASSICAL COMMUNICATION

The next issue to consider is that of the classical communication required. At stage l the set of entangled resource states is of the form

$$|\Psi_l\rangle = [\cos(\beta_l)|0\rangle^{\otimes N} + i \sin(\beta_l)|1\rangle^{\otimes N}]^{\otimes M_l}, \quad (11)$$

where M_l is the number of entangled states used in stage l , and is equal to $\lfloor Mp(l)(1+\delta) \rfloor$. We may alternatively express this state as

$$|\Psi_l\rangle = \sum_{\mathbf{i}} \mu_{\mathbf{i}} |\mathbf{i}\rangle, \quad (12)$$

where $\mathbf{i} = (i_1, \dots, i_{M_l})$, $\mu_{\mathbf{i}} = \prod_m \sin^{i_m}(\beta_l) \cos^{1-i_m}(\beta_l)$ and $|\mathbf{i}\rangle = |i_1\rangle \otimes \dots \otimes |i_{M_l}\rangle$.

Now we make an approximation to the state $|\Psi_l\rangle$ by retaining only typical sequences of the i . Denoting the set of typical \mathbf{i} by S_l , the approximate state is

$$|\tilde{\Psi}_l\rangle \propto \sum_{\mathbf{i} \in S_l} \mu_{\mathbf{i}} |\mathbf{i}\rangle^{\otimes N}. \quad (13)$$

It is a standard result for typical sequences that, for all $\epsilon, \delta > 0$, there exists an M_l such that the fidelity is at least $1 - \epsilon/L$ with the number of elements in S_l no more than $2^{M_l E(\beta_l)(1+\delta)}$.

Because fidelity does not decrease under completely-positive trace-preserving (CPTP) maps, the fidelity of the final state with the output for the exact state must be at least $1 - \epsilon/L$. In the case where we use approximate resource states with typical sequences at each stage, the final output must have fidelity at least $1 - \epsilon$ with the state obtained with the exact resource states, ρ . The fidelity with the desired state, $|\phi\rangle$, is then at least $1 - 2\epsilon$.

Now, to reduce the average communication required, we replace Step 2 with a joint measurement in the Fourier transform basis on the states $|\mathbf{i}\rangle$ for typical sequences by parties 1 to $N - 1$. The communication required for the measurement result does not exceed $M_l E(\beta_l)(1 + \delta)$ bits. Using the expression for M_l , this does not exceed $Mp(l)E(\beta_l)(1 + \delta)^2$.

There are a number of different ways this communication may be performed. The communication may be performed from party 1 to 2, then the sum (modulo $\|S_l\|$) communicated to party 3, and so forth up to the communication to party N . Alternatively the communication may be performed directly from each party (1 to $N - 1$) to party N . Many other combinations of communication are possible, provided it is possible for party N to determine the sum (modulo $\|S_l\|$) of the measurement results.

After this communication is performed, we simply perform Step 4 as above. Now it is necessary to communicate the positions of the measurement successes from party N to the other parties. It does not appear to be possible to reduce this communication via the typical sequences approach as it does for the other communication.

The average total communication from each party 1 to $N - 1$ does not exceed $\sum_l p(l)E(\beta_l)(1+\delta)^2 = \mathcal{E}(\alpha)(1+\delta)^2$.

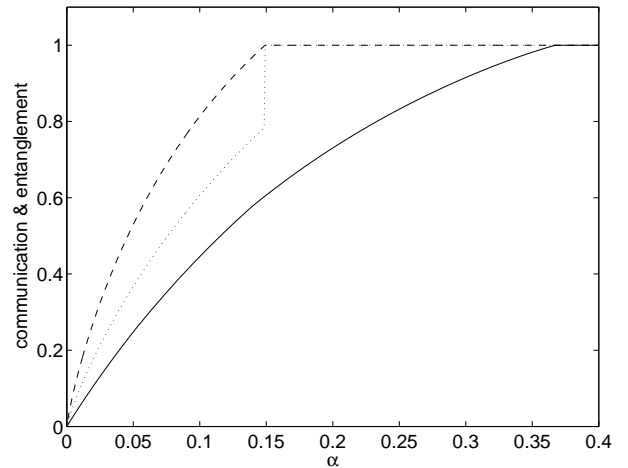


FIG. 3: Results for schemes optimised for minimum classical communication from the last party. This communication is shown as the dashed line, the corresponding entanglement required is shown as the dotted line. The entanglement for the entanglement-optimised scheme is shown as the solid line for comparison.

Thus, in the limit of small δ , this communication is the same as the entanglement consumed. The communication from party N will, in general, be much larger, and does not scale down with α . For example, in Fig. 1 the communication required is over 9 bits for the smallest (nonzero) value of α plotted.

It is also possible to select the entangled resources to minimise the average communication required from the last party. In this case numerical results indicate that the optimal scheme uses an entangled resource with $\tan(\beta) = \tan^2(\alpha)$. For small α , the probability of failure is small. In the case of failure we simply implement the scheme with 1 ebit of entanglement and 1 bit of communication in each direction. The corresponding classical communication is shown in Fig. 3.

The classical communication is still larger than the entanglement for the scheme optimised for minimum entanglement consumption. However, it does go to zero in the limit of small α . Unfortunately the ratio of the classical communication to α is not bounded for small α (see Fig. 4). The ratio increases approximately as $|\log \alpha|$. This means that a scheme based on implementing a series of operations of the form $U(\alpha)$ will still require a large amount of communication from the last party.

V. GENERAL TENSOR PRODUCT HAMILTONIANS

In the previous sections we have shown how to implement evolution under a Hamiltonian of the form $\sigma_z^{\otimes N}$. Now we show how to implement a general tensor product Hamiltonian of the form $H = H_1 \otimes \dots \otimes H_N$. To do this, we apply a method similar to that in Ref. [9]. First

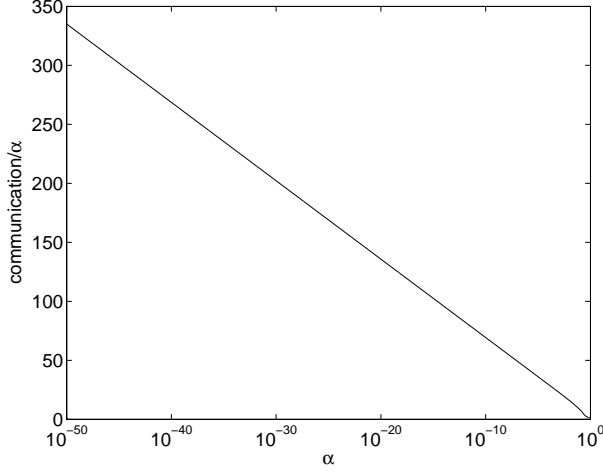


FIG. 4: The ratio of the communication from the final party to the interaction strength α .

note that it is possible to diagonalise the H_j via local unitaries. Taking $\Delta = \|H\|$, the diagonalised form of H is

$$H_{\text{diag}} = \Delta \bigotimes_j \text{diag}(a_{1,j}, a_{2,j}, a_{3,j}, \dots, a_{d_j-1,j}, a_{d_j,j}). \quad (14)$$

where $a_{l,j} \in [-1, 1]$, and d_j is the dimension of the subsystem which H_j acts upon. In the following we consider simulation of this diagonalised form.

This Hamiltonian may be simulated by the Hamiltonian $\sigma_z^{\otimes N_k}$ in a similar way as in Ref. [9]. The chain of simulations used is

$$\begin{aligned} \sigma_z^{\otimes N} &\rightarrow H''_{A_1 B_1} \otimes \sigma_z^{\otimes N-1} \rightarrow H''_{A_1 B_1} \otimes H''_{A_2 B_2} \otimes \sigma_z^{\otimes N-2} \\ &\dots \rightarrow \bigotimes_j H''_{A_j B_j} \rightarrow H_{\text{diag}}/\Delta, \end{aligned} \quad (15)$$

where $H''_{A_j B_j} = \text{diag}(a_{1,j}, -a_{1,j}, \dots, a_{d_j,j}, -a_{d_j,j})$. We use the A_j subscripts to indicate the subsystems upon which we wish to implement H , and B_j to indicate ancilla subsystems.

In order to perform this chain of simulations, we need to, in general, perform the simulation

$$\bigotimes_{n=1}^{j-1} H''_{A_n B_n} \otimes \sigma_z^{\otimes (N-j+1)} \rightarrow \bigotimes_{n=1}^j H''_{A_n B_n} \otimes \sigma_z^{\otimes (N-j)}. \quad (16)$$

To do this, we first append the d_j dimensional ancilla B_j so the j 'th term in the tensor product is $\text{diag}(1, -1, 1, -1, \dots, 1, -1)$. Now we take $p_l = (a_{l,j} + 1)/2 \in [0, 1]$, and define the local unitaries U_l which exchange the $(2l-1)$ -th and $(2l)$ -th basis vectors of $A_j B_j$. To simulate $\bigotimes_{n=1}^j H''_{A_n B_n} \otimes \sigma_z^{\otimes (N-j)}$ for small time δt , we simply apply U_l at time $p_l \delta t$ and again at time δt . In this way, we may apply each successive simulation in the chain (15). The final simulation in the chain may

be achieved by simply restricting to the appropriate subspace.

Ref. [9] gave a similar method for the case of bipartite tensor product Hamiltonians (though the multipartite case was mentioned briefly in the discussion). In the bipartite case of Ref. [9], the simulation is reversible. This is because the diagonal Hamiltonian may be expanded as a sum of local Hamiltonians and a Hamiltonian for which the maximum and minimum eigenvalues have the same magnitude. The local Hamiltonians may be ignored, because they can be implemented locally. One can then use H_{diag}/Δ to simulate $\sigma_z^{\otimes N}$ by restricting to the subspace for the maximum and minimum eigenvalues. In the multipartite case we can not use a similar method, because the additional terms will be multipartite Hamiltonians on $N-1$ subsystems, rather than local Hamiltonians.

The case of self-inverse Hamiltonian evolution, as in Ref. [6], is particularly simple. All the $a_{l,j}$ are ± 1 , so it is possible to achieve the entire simulation chain without intermediate unitaries simply by restricting to the appropriate subspaces. Thus self-inverse Hamiltonian evolution is equivalent to evolution under $\sigma_z^{\otimes N}$.

In order to simulate evolution under H for a time interval t , we divide the time interval into m intervals of length t/m . For m sufficiently large, the chain of simulations above will be accurate. The expectation value for the consumed entanglement is then no more than $5.6418t\|H\|$. In fact, the expectation value is approximately equal to this, because each simulation of $U(\alpha)$ is for $\alpha \ll 1$. As above, we may also obtain an asymptotic average entanglement consumed equal to this value in the limit of a large number of implementations of H . Also the asymptotic average communication from each of the parties $1 \dots N-1$ may be made equal to this value.

Note that it is necessary to consider a large number of implementations of the Hamiltonian for small time intervals, because in addition to the division of t into m intervals in order to make the simulations in Eq. (15) accurate, the simulations in Eq. (15) require further subdivision of the time. However, for a given fidelity $1 - \epsilon$ required, the number of subintervals is fixed, so there is no problem in taking the appropriate limits to obtain the average entanglement and communication.

We can also use the above methods to simulate more general Hamiltonians of the form $H = \sum_k H_k$, where each H_k is a tensor product Hamiltonian. We simply implement $(\prod_k e^{-itH_k/m})^m$. For large m , this simulation may be made arbitrarily accurate. The average entanglement consumed is approximately $5.6418t \sum_k \|H_k\|$. For completely general Hamiltonians a large number of terms in the sum will be required, so this method will not be efficient. However, it should be useful for cases with only a moderate number of terms required.

VI. FURTHER GENERALISATION

We may generalise the above methods to general unitaries of the form

$$U = \sum_k \lambda_k V_k^{(1)} \otimes V_k^{(2)} \otimes \dots \otimes V_k^{(N)}, \quad (17)$$

where the $V_k^{(j)}$ are local unitaries, and λ_k is small except for the k such that $V_k^{(j)} = \mathbb{1}$. To implement this operation, we use an entangled resource state of the form

$$|\Psi_U\rangle = \sum_k \mu_k |k\rangle_{B_1} \otimes \dots \otimes |k\rangle_{B_N}. \quad (18)$$

We use B_j for the subsystems for the resource state, and A_j for the subsystems for the system state (which we wish to apply U to).

Each party j performs the controlled operation $\sum_k |k\rangle_{B_j} \langle k| \otimes V_k^{(j)}$, where $V_k^{(j)}$ acts upon A_j . As a result, we have the stator

$$\sum_k \mu_k |k\rangle_{B_1} \otimes V_k^{(1)} \otimes \dots \otimes |k\rangle_{B_N} \otimes V_k^{(N)} \quad (19)$$

acting upon the system.

Each party performs local measurements, and communicates the results to party N , where the phase correction is made. We then perform a projective measurement on the resource state in subsystem B_N , with success corresponding to projection onto the state $\sum_k \nu_k |k\rangle$. Choosing $\mu_k \nu_k^* \propto \lambda_k$, U is implemented in the case of success.

In the case of failure a different unitary operation is performed. This case may be corrected for by implementing the correction in the same way, or alternatively by teleporting the portions of the state in subsystems A_1 to A_{N-1} to party N , performing the appropriate unitary, then teleporting these portions of the state back to their respective subsystems.

For example, we may use this approach with two-qubit unitaries. Two-qubit unitaries may be simplified to the form [10, 11]

$$U = \sum_{k=0}^3 \lambda_k \sigma_k \otimes \sigma_k, \quad (20)$$

where we take the convention that $\sigma_0 = \mathbb{1}$. In the case of the incorrect measurement result, the error operation is again of this form, and we wish to implement an operation of this form for the correction. Repeating the process, we may implement the unitary in a more direct way than by implementing a sequence of unitaries of

the form $U(\alpha)$. However, calculations indicate that this method consumes more entanglement.

One use of this approach would be to implement the operation with small classical communication in all directions. As discussed at the end of Sec. IV, it is possible to optimise for minimum communication from the last party, however the ratio of the classical communication to α is not bounded. This means that, if we use this method for more general tensor product Hamiltonians, the average communication from the last party is still going to be large. On the other hand, using the approach given in this section, we can ensure that the average communication required approaches zero in the limit as U approaches the identity. By selecting $\mu_k \propto \nu_k \propto \sqrt{\lambda_k}$, we find that the probability of failure approaches zero as U approaches the identity.

VII. CONCLUSION

We have presented a scheme for implementing evolution under general multipartite tensor product Hamiltonians. This scheme requires multipartite entanglement of approximately $5.6418t\|H\|$. In contrast, the scheme of CDKL [3], which applies to the $\sigma_x \otimes \sigma_x$ interaction, requires entanglement of $5.9793t\|H\|$. This scheme may be applied both to the case of a single implementation, in which case this is the expectation value of the entanglement consumption, or to multiple implementations.

In the case of multiple implementations, the entanglement consumption is the average over the implementations, given that the implementations are achieved with arbitrarily high fidelity. In addition, for multiple implementations the average communication in some of the directions may be reduced to be equal to the entanglement consumption. However, a large amount of communication from one of the subsystems is required.

This scheme may also be applied to more general Hamiltonians. A general Hamiltonian may be expressed as a sum of tensor product Hamiltonians, and therefore may be implemented using the above approach together with a Trotter expansion. The drawback to this approach is that it will be inefficient if a large number of terms is required. It is also possible to apply this approach to CPTP maps which are close to the identity. By adding an ancilla, the map may be replaced with a unitary which is close to the identity, which may be implemented in the above way. Again this method will be inefficient if a large number of terms are required.

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